Combinatorial Networks
Week 5, Wednesday

Turán's Theorem

- **Definition.** Let H be a fixed graph, we say G is *H-free* if G has NO any copy of H.
- **Definition.** $ex(n,H)$ is the maximal number of edges in n-vertices H-free graphs.
- **Mental's theorem.** $ex(n,K_3) \le \lfloor \frac{n^2}{4} \rfloor$, and the unique extremal graph is $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.
- **Theorem.** $ex(n,K_{2,2}) \le \frac{1}{2}(n^{\frac{3}{2}}+n)$.
- **Definition.** *Turán graph* $T_r(n)$ is a complete r-partite graph, where the sizes of parts differ by at most 1.
- **Note.** $T_2(n) = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.
- **Turán's Theorem.** (1941) $T_r(n)$ is the unique K_{r+1}-free graph on n vertices with the maximum number of edges.
  Thus $ex(n,K_{r+1}) \approx (1-\frac{1}{r})\frac{n^2}{2}$ (with equality if and only if $r|n$).
- **Proof 1 of Turán's Theorem.** By induction on r.
  Base case, $r=2$, which is Mental's theorem.
  Inductive step, assume it is true for K_r, consider a K_{r+1}-free graph G with n vertices and the maximal number of edges. Pick a vertex u with maximum degree $d=d(u)$, let $S=N(u)$, $T=V-S$, then S is K_r-free on d vertices. Let G' be obtained from G by deleting all edges in T and adding all missing edges across S and T, then G' is still K_{r+1}-free.
  $$e(G) \le e(G').$$
  We claim: $e(G) \ge e(G')$.
  $e(G') = e(G) - e(T) + \#$missing edges between S and T in G.
  (1) $\sum_{t\in T} d(t) = 2e(T) + e(S,T) \le d(T)$.
  (2) #missing edges between S and T in $G = |S||T| - e(S,T)$. so
  $$\begin{aligned} e(G') &= e(G) - e(T) + d(T) - e(S,T) \\ &\ge e(G) - e(T) + 2e(T) + e(S,T) - e(S,T) \\ &= e(G) + e(T) \\ &\ge e(G) \end{aligned}$$
  Thus $e(G) = e(G'), e(T) = 0$ and for any $t \in T, d(t) = d(u)$, $e(G) = e(S) + |S||T| \le e(T_{r-1}(d)) + d(n-d)$, this function is maximized when $d \approx \frac{r-1}{r}n$ or say the r parts are of size differ by at most 1, so $G = T_r(n)$. ∎
- **Proof 2 of Turán's Theorem.** Consider K_{r+1}-free graph G on n vertices, label the n vertices as $1,2,...,n$ and consider a probability p_i for each i such that $\sum p_i = 1$ and $p_i \ge 0$, change the values of p_i to maximize
  $$P = \sum_{ij\in E(G)} p_i p_j.$$

Claim: If $ij \notin E(G)$ and $p_i, p_j > 0$ then one can change

$$\begin{cases} p_i \to 0 \\ p_j \to p_i + p_j \end{cases}$$

or

$$\begin{cases} p_i \to p_i + p_j \\ p_j \to 0 \end{cases}$$

to increase P.
Proof of claim. Let $S_i = \sum_{k \in N(i)} p_k, S_j = \sum_{k \in N(j)} p_k$, we assume that $S_i \geq S_j$, change

$$\begin{cases} p_i \to p_i + p_j \\ p_j \to 0 \end{cases}$$

then $P' - P = p_j S_i - p_j S_j = p_j(S_i - S_j) \geq 0$. ∎
Now we keep applying the above claim. When stop, we arrive on vector $(p_1, p_2, ..., p_n)$ such that the vertices i (with $p_i > 0$) form a clique Q. Since G is K_{r+1}-free, we have $|Q| \leq r$. Consider

$$P = \sum_{ij \in E(G)} p_i p_j = \frac{1}{2}(1 - \sum_{i \in V(Q)} p_i^2),$$

this achieves maximum if and only if $p_i = \frac{1}{|Q|}$ for any $i \in Q$, $P = \frac{1}{2}(1 - \frac{1}{|Q|}) \leq \frac{1}{2}(1 - \frac{1}{r})$.
But when $p_1 = p_2 = ... + p_n = \frac{1}{n}, P_0 = \frac{e(G)}{n^2}$, thus $\frac{e(G)}{n^2} \leq \frac{1}{2}(1 - \frac{1}{r})$, which means $e(G) \leq \frac{n^2}{2}(1 - \frac{1}{r})$. ∎

- **Definition.** An *independent set* is a subset of $V(G)$ which induces NO edges in G.
- **Definition.**

$$\alpha(G) = \max_{I = \text{independent set in } G} |I|.$$

- **Theorem.** For any $G = (V, E)$, $\alpha \geq \sum_{v \in V} \frac{1}{d(v)+1}$.
Proof. Label vertices as $1, 2, ..., n$, consider a permutation π on $[n]$.
We say vertex i is π-dominating, if for any $j \in N(i), \pi(i) < \pi((j)$, i.e. i precedes all its neighbors in the ordering given by permutation π. Let $M_\pi = \{\text{all } \pi\text{-dominating vertices } i\}$.
Fact 1. M_π is an independent set of G.
Fact 2. $\alpha(G) \geq |M_\pi|$ for any π.
Consider a permutation π uniformly at random, thus, $|M_\pi|$ is a random variable and by fact 2, we have $\alpha(G) \geq \mathbb{E}[|M_\pi|]$.
Let A_i be the event that $i \in M_\pi$, so

$$I_{A_i}(\pi) = \begin{cases} 1, i \in M_\pi \\ 0, \text{otherwise.} \end{cases}$$

Then $|M_\pi| = \sum_{i \in V(G)} I_{A_i}, \mathbb{E}[|M_\pi|] = \sum_{i \in V(G)} Pr(A_i)$.

Fact 3. $Pr(A_i) = \frac{1}{d(i)+1}$.
Proof of fact 3.

$$
\begin{aligned}
Pr(A_i) &= Pr(i \text{ is } \pi\text{-dominating}) \\
&= Pr(\pi(i) \text{ is minimum in } i \cup N(i)) \\
&= \frac{1}{d(i)+1}
\end{aligned}
$$

∎

Therefore, $\alpha(G) \geq \mathbb{E}[|M_\pi|] = \sum_{i \in V(G)} Pr(A_i) = \frac{1}{d(i)+1}$. ∎

- **Corollary.** If G has n vertices and m edges, then

$$\alpha(G) \geq \frac{n^2}{2m+n}.$$

Proof. Exercise.

- **Proof** 3 **of Turán's Theorem.** For K_{r+1}-free graph G, consider its complement G^c with $m = \binom{n}{2} - e(G)$ edges,

$$r \geq \alpha(G^c) \geq \frac{n^2}{2m+n} = \frac{n^2}{n - 2e(G)}$$

which means $e(G) \leq (1 - \frac{1}{r})\frac{n^2}{2}$. ∎